§ Proof of Gauss-Bonnet Theorems

We will prove Gauss-Bonnet Theorem I, i.e.

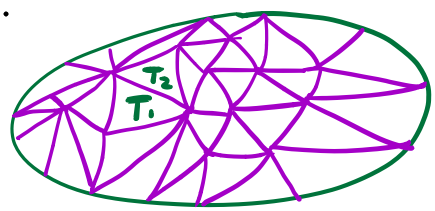
$$\boxed{\int_S K \, da = 2\pi \chi(S)}$$

for any compact orientable surface S without boundary

First, we reduce the problem to the "local" case by

<u>FACT</u>: Any orientable surface can be "triangulated".

e.g.



$$S = \bigcup_i T_i$$

By subdivision we can assume that each T_i is small enough that it is contained in a single coordinate neighborhood.

<u>FACT</u>: Any compact orientable surface can be covered by "conformal coordinate systems":

i.e. $X : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S$

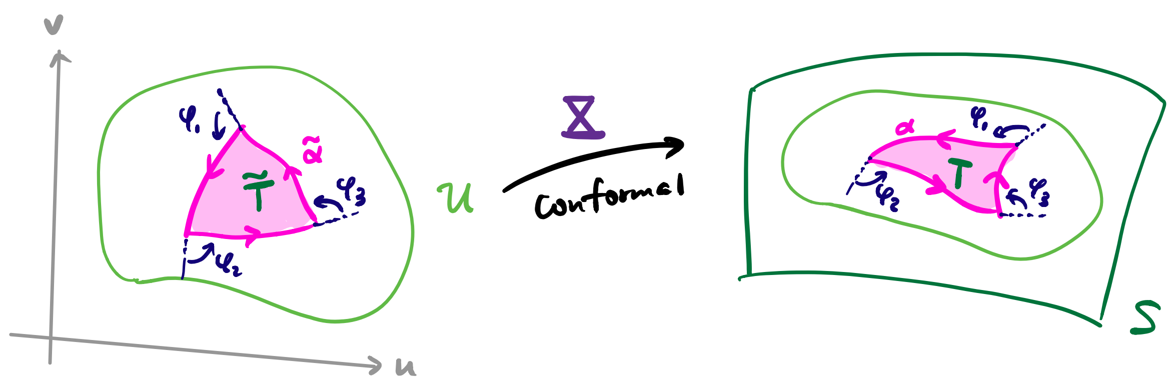
s.t. $\underbrace{(g_{ij})}_{1^{st}\text{ f.f.}} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for some smooth positive function $\lambda : \mathcal{U} \to \mathbb{R}$.

Note: Conformal coordinates preserve infinitesimal angles!

We will prove the following local Gauss-Bonnet Theorem:

Local Gauss-Bonnet:

Let $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S$ be a conformal parametrization.



$$\boxed{\int_T K \, da + \int_{\partial T} k_g \, ds + \sum_i \varphi_i = 2\pi}$$

Proof: In conformal coordinates,

$$k_g = \frac{1}{2\lambda}\left(\frac{\partial \lambda}{\partial u} v' - \frac{\partial \lambda}{\partial v} u'\right) + k$$

where k = curvature of $\tilde{\alpha}$ (as a plane curve in $\mathbb{R}^2$)

$\tilde{\alpha}(s) = (u(s), v(s))$ p.b.a.l. as a plane curve

$$\int_{\partial T} k_g \, ds + \sum_i \varphi_i = \int_{\tilde{\alpha}} \frac{1}{2\lambda}\left(\frac{\partial \lambda}{\partial u} v' - \frac{\partial \lambda}{\partial v} u'\right) + \underbrace{\int_{\tilde{\alpha}} k + \sum_i \varphi_i}_{= 2\pi \text{ (Thm. of Turning tangent)}}$$

By Green's Theorem,

$$\int_{\tilde{\alpha}} \frac{1}{2\lambda^2}\left(\frac{\partial \lambda^2}{\partial u} v' - \frac{\partial \lambda^2}{\partial v} u'\right) = \iint_{\tilde{T}} \frac{\partial}{\partial u}\left(\frac{1}{2\lambda}\frac{\partial \lambda}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{2\lambda}\frac{\partial \lambda}{\partial v}\right)$$

$$= \iint_{\tilde{T}} \frac{1}{2} \Delta(\log \lambda) \, du\, dv$$

$$= \iint_{\tilde{T}} \underbrace{\frac{1}{2\lambda} \Delta(\log \lambda)}_{-K} \underbrace{\lambda \, du\, dv}_{da}$$

$$= -\int_T K \, da$$

$\square$

We now apply local Gauss-Bonnet to each T_i in a "fine" triangulation with

$$\begin{cases} F = \#\text{ faces} \\ E = \#\text{ edges} \\ V = \#\text{ vertices} \end{cases}$$

Recall:

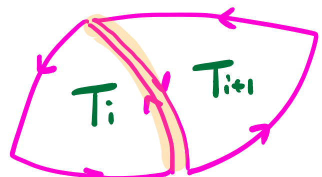
$$\chi(S) = F - E + V$$

$$\int_{T_i} K\, da + \int_{\partial T_i} k_g\, ds + \sum_j \varphi_j^i = 2\pi$$

Sum over all the T_i's, we get

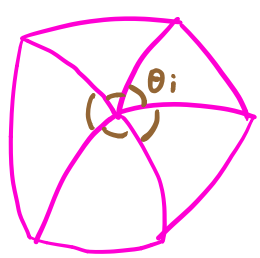
$$\int_S K\, da + 0 + \begin{matrix} 3\pi F \\ -2\pi V \end{matrix} = 2\pi F$$

Reason: (I)



different orientations $\Rightarrow$ cancels

(II)



ext. $\angle$ sum $= 3\pi -$ int. $\angle$ sum

$$\Downarrow$$

$$\sum_{i,j} \varphi_j^i = 3\pi F - 2\pi V$$

Hence,

$$\int_S K da = 2\pi F - \underbrace{3\pi F}_{= 2\pi E} + 2\pi V$$

since $3F = 2E$

$\blacksquare$